Physical and Mathematical Models

By:

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Static Physical Models

- The best known examples of physical models are scale models.
- In shipbuilding, making a scale model provides a simple way of determining the exact measurements of the plates covering the hull, rather than having to produce drawings of complicated, three-dimensional shapes.

Static Physical Models

- Scientists have used models in which spheres represent atoms, and rods or specially shaped sheets of metal connect the spheres to represent atomic bonds.
- Scale models are also used in wind tunnels and water tanks in the course of designing aircraft and ships.

Static Physical Models

- Although air is blown over the model, or the model is pulled through the water, these are static physical models because the measurements that are taken represent attributes of the system being studied under one set, equilibrium conditions.
- In this case, the measurements do not translate directly into system attribute values.

Static Physical Models

- Well known laws of similitude are used to convert measurements on the scale model to the values that would occur in the real system.
- Sometimes, a static physical model is used as a means of solving equations with particular boundary conditions.
- There are many examples in the field of mathematical physics where the same equations apply to different physical phenomena.

Static Physical Models

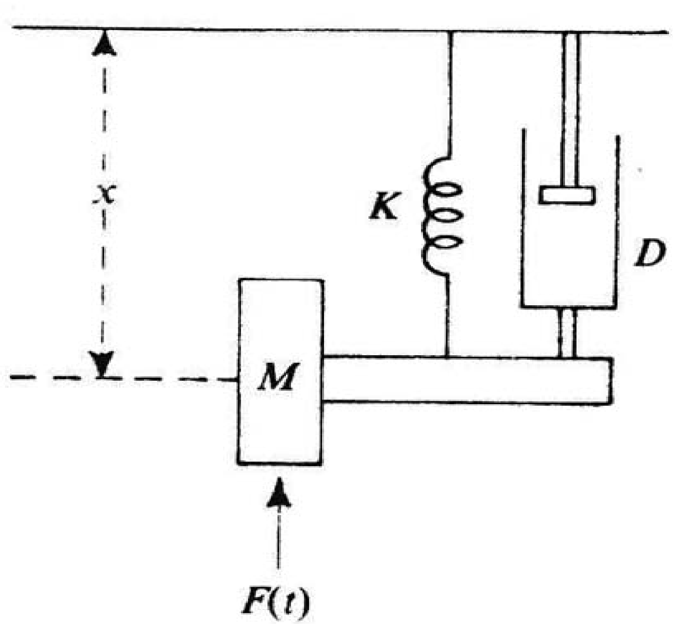
- For example, the flow of heat and the distribution of electric charge through space can be related by common equations.

Dynamic Physical Models

- Dynamic physical models rely upon an analogy between the system being studied and some other system of a different nature, the analogy usually depending upon an underlying similarity in the forces governing the behavior of the systems.
- To illustrate this type of physical model, consider the two systems shown in following figures i.e. Figure 2 and Figure 3.

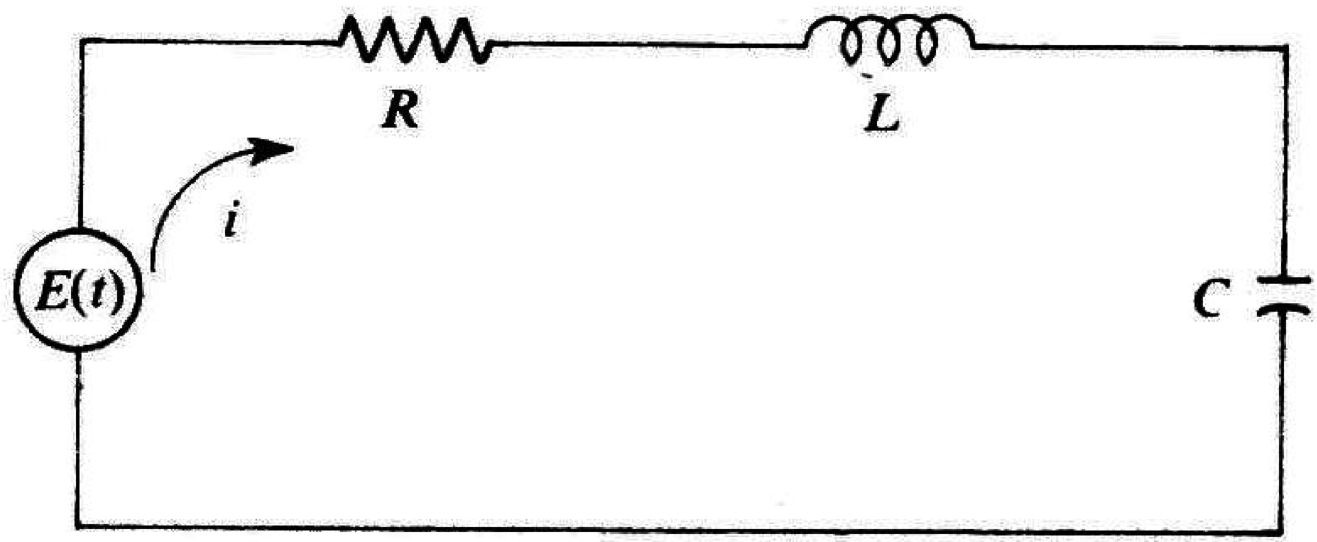
Dynamic Physical Models

- Figure 2. Mechanical System



Dynamic Physical Models

- Figure 3. Electrical System



Dynamic Physical Models

- The Figure 2. represents a mass that is subject to an applied force F(t) varying with time, a spring whose force is proportional to its extension or contraction, and a shock absorber that exerts a damping force proportional to the velocity of the mass.
- The system might for example represent the suspension of an automobile wheel when the automobile body is assumed to be immobile in a vertical direction.

Dynamic Physical Models

- It can be shown that the motion of the system is described by the following differential equation.

$$M\ddot{x} + D\dot{x} + Kx = KF(t)$$

Where,

x is the distance moved,

M is the mass,

K is the stiffness of the spring,

D is the damping factor of the shock absorber

Dynamic Physical Models

- Figure 3. represents an electrical circuit with an inductance L, a resistance R, and a capacitance C, connected in series with a voltage source that varies in time according to the function E(t). If q is the charge on the capacitance, it can be shown that the behavior of the circuit is governed by the following differential equation:

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = \frac{E(t)}{C}$$

$\dot{q}$

Dynamic Physical Models

- Inspection of these two equations shows that they have exactly the same form and that the following equivalences occur between the quantities in the two systems:

Dynamic Physical Models

Displacement x	Charge q
Velocity $\dot{x}$	Current $I(=\dot{q})$
Force F	Voltage E
Mass M	Inductance L
Damping factor D	Resistance R
Spring stiffness K	1/Capacitance $1/C$

Dynamic Physical Models

- The mechanical system and the electrical system are analogs of each other, and the performance of either can be studied with the other.
- In practice, it is simpler to modify the electrical system than to change the mechanical system, so it is more likely that the electrical system will have been built to study the mechanical system.

Dynamic Physical Models

- If, for example, a car wheel is considered to bounce too much with a particular suspension system, the electrical model will demonstrate this fact by showing that the charge (and, therefore, the voltage) on the condenser oscillates excessively.
- To predict what effect a change in the shock absorber or spring will have on the performance of the car, it is only necessary to change the values of the resistance or condenser in the electrical circuit and observe the effect on the way the voltage varies.

Dynamic Physical Models

- If in fact, the mechanical system were as simple as illustrated, it could be studied by solving the mathematical equation derived in establishing the analogy.
- However, effects can easily be introduced that would make the mathematical equation difficult to solve.

Static Mathematical Model

- A static model gives the relationships between the system attributes when the system is in equilibrium.
- If the point of equilibrium is changed by altering any of the attribute values, the model enables the new values for all the attributes to be derived but does not show the way in which they changed to their new values.

Static Mathematical Model

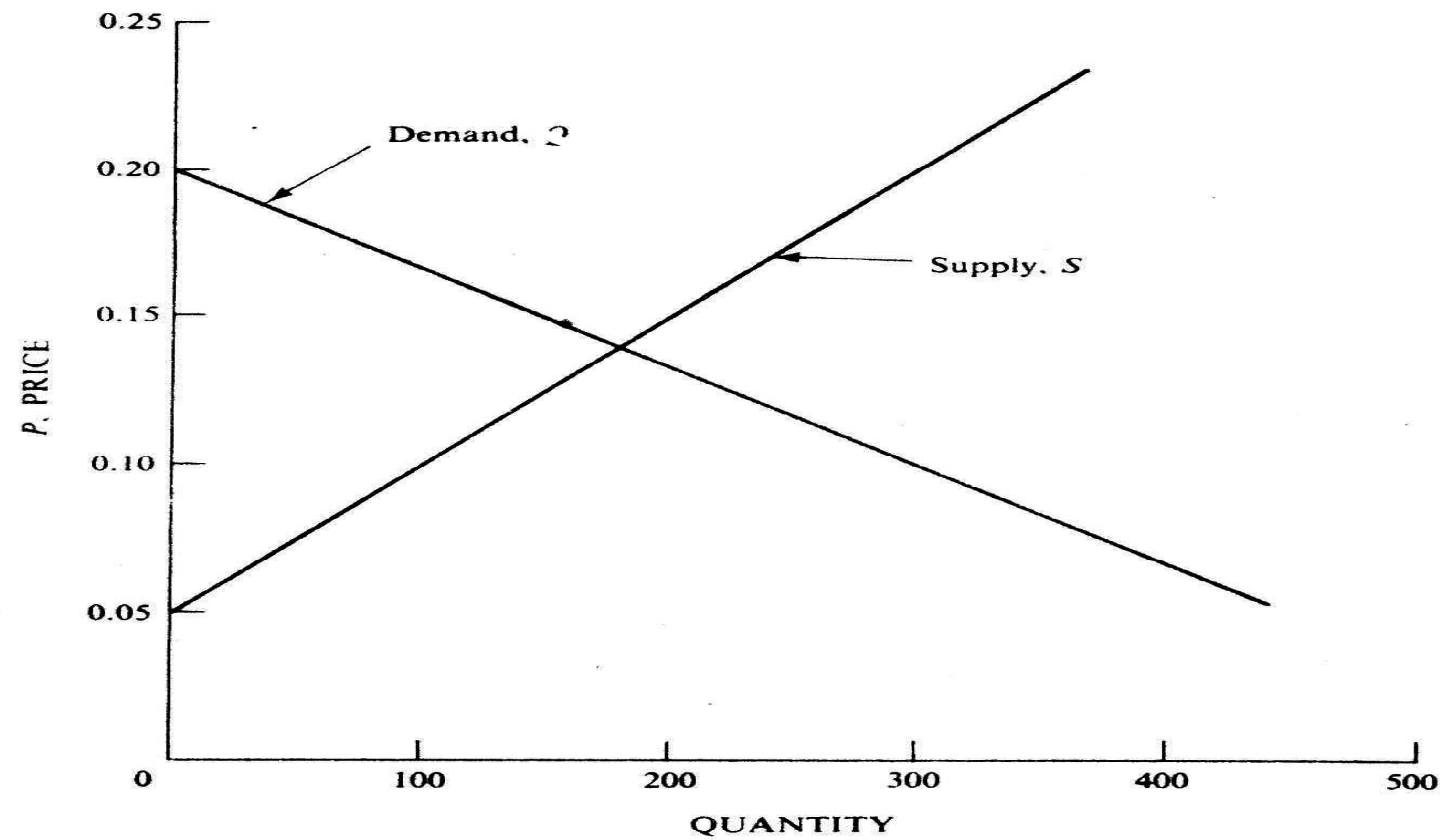
- For example, in marketing a commodity there is a balance between the supply and demand for the commodity.
- Both factors depend upon price: a simple market mode! will show what is the price at which the balance occurs.
- Demand for the commodity will be low when the price is high, and it will increase as the price drops.

Static Mathematical Model

- The relationship between demand, denoted by Q, and price, denoted by P, might be represented by the straight line marked "Demand“ in Figure 4.
- On the other hand, the supply can be expected to increase as the price increases, because the suppliers see an opportunity for more revenue.
- Suppose supply, denoted by S, is plotted against price, and the relationship is the straight line marked "Supply" in Figure 4.lf conditions remain stable, the price will settle to the point at which the two lines cross, because that is where the supply equals the demand.
- Since the relationships have been assumed linear, the complete market model can be written mathematically as follows:

Static Mathematical Model

- Figure 4. Linear Market Model



Static Mathematical Model

$Q=a-bP$

$S=c+dP$

$S=Q$

- The last equation states the condition for the market to be cleared; it says supply equals demand and, so, determines the price to which the market will settle.
- For the model to correspond to normal market conditions in which demand goes down and supply increases as price goes up the coefficients b and d need to be positive numbers.

Static Mathematical Model

- For realistic, positive results, the coefficient a must also be positive. Figure 4 has been plotted for the following values of the coefficients:

  a=600

  b=3000

  c=-100

  d=2000

Static Mathematical Model

- The fact that linear relationships have been assumed allows the model to be solved analytically. The equilibrium market price, in fact, is given by the following expression:

$$p = \frac{a - c}{b + d}$$

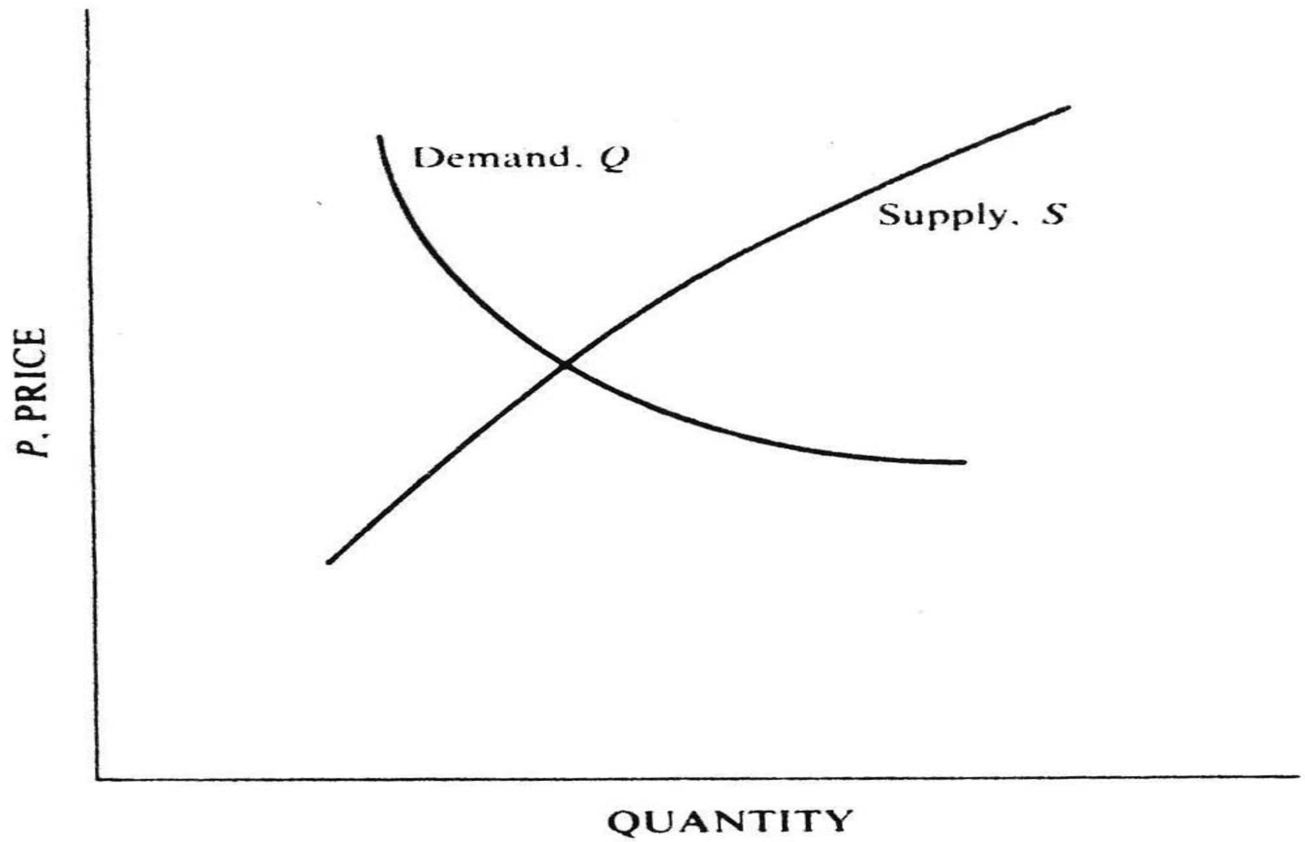
- With the chosen values, the equilibrium price is 0.14, which corresponds to a supply of 180.

Static Mathematical Model

- More usually, the demand will be represented by a curve that slopes downwards, and the supply by a curve that slopes upwards, as illustrated in Figure. 5. It may not then be possible to express the relationships by equations that can be solved.
- Some numeric method is then needed to solve the equations.

Static Mathematical Model

- Figure 5. Non linear market model



Static Mathematical Model

- Drawing the curves to scale and determining graphically where they intersect is one such method.
- In practice, it is difficult to get precise values for the coefficients of the model.
- Observations over an extended period of time, however, will establish the slopes (that is, the values of b and d) in the neighborhood of the equilibrium point, and, of course, actual experience will have established equilibrium prices under various conditions.

Static Mathematical Model

- The values depend upon economic factors, so the observations will usually attempt to correlate the values with the economy, allowing the model to be used as a means of forecasting changes in market conditions for anticipated economic changes.

Dynamic Mathematical Model

- A dynamic mathematical model allows the changes of system attributes to be derived as a function of time.
- The derivation may be made with an analytical solution or with a numerical computation, depending upon the complexity of the model.
- The equation that was derived to describe the behavior of a car wheel is an example of a dynamic mathematical model; in this case, an equation that can be solved analytically.

Dynamic Mathematical Model

- It is customary to write the equation in the form

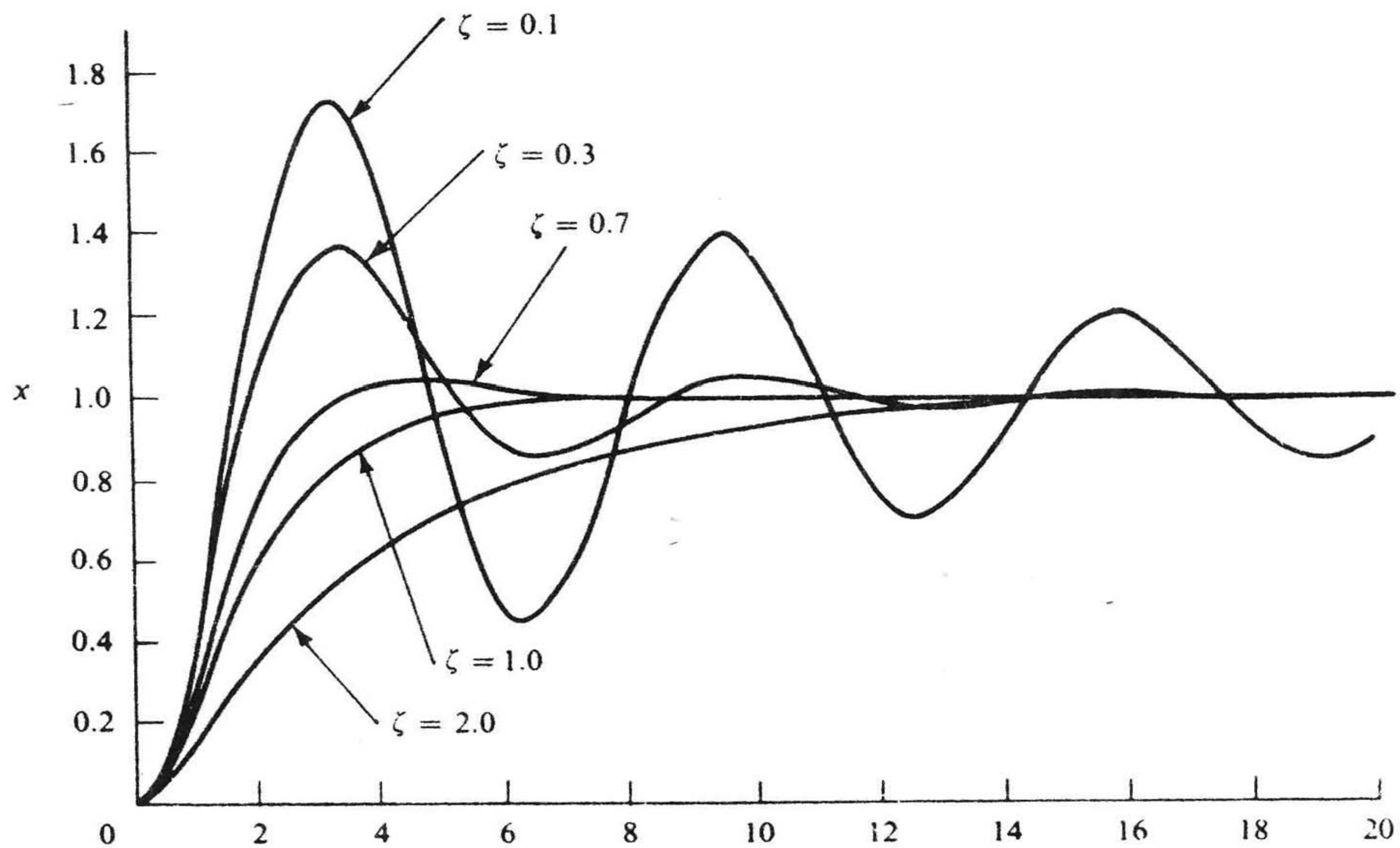
$$\ddot{x} + 2\zeta\omega + \omega^2 = \omega^2 F(t)$$

Where $2\zeta\omega = D/M$ and $\omega^2 = K/M$

- Expressed in this form, solutions can be given in terms of the variable wt. Figure.6 shows how x varies in response to a steady force applied at time t = 0 as would occur, for instance, if a load were suddenly placed on the automobile.
- Solutions are shown for several values of ζ, and it can be seen that when ζ is less than 1, the motion is oscillatory.

Dynamic Mathematical Model

- Figure 6. Solutions of second order equations



Dynamic Mathematical Model

- The factor ζ is called the damping ratio and, when the motion is oscillatory, the frequency of oscillation is determined from the formula.

$$\omega = 2\pi f$$

- Where f is the number of cycles per second.
- Suppose a case is selected is representing a satisfactory frequency and damping. The relationship given above between ζ , ω ,M, k and D show how to select the spring and shock absorber to get that type of motion. For example the condition for the motion to occur without oscillation requires that $\zeta \geq 1$. It can be deduced from the definition of that the condition requires that $D^2 \geq 4MK$.

Reference

- Geoffrey Gordon, System Simulation